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Dynamic Solution of the HJB Equation and the Optimal Control of Nonlinear Systems

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Abstract— **Optimal control problems are often solved exploiting the solution of the so-called Hamilton-Jacobi-Bellman (HJB) partial differential equation, which may be, however, hard or impossible to solve in specific examples. Herein we circumvent this issue determining a *dynamic* solution of the HJB equation, without solving any partial differential equation. The methodology yields a dynamic control law that minimizes a cost functional defined as the sum of the original cost and an additional cost.**

I. Introduction

Optimal control deals with the problem of finding a control law such that the origin is an asymptotically stable equilibrium point of the closed-loop system and moreover a given criterion is optimized. An optimal control policy may be given by a set of differential equations describing the paths of the control variable that minimize the cost functional which is in general a function of the state and of the control input.

Consider a nonlinear system, affine in the control, described by an equation of the form

$$\dot{x} = f(x) + g(x)u\,, \tag{1}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ smooth mappings, where $x(t) \in \mathbb{R}^n$ denotes the state of the system and $u(t) \in \mathbb{R}^m$ the input. The task of the control is to minimize the cost functional

$$J(x(t), u(t)) = q_{t_f}(x(t_f)) + \int_0^{t_f} L(x(t), u(t))dt\,, \tag{2}$$

where $L(\cdot,\cdot)$ is positive semidefinite, subject to the nonlinear first-order dynamic constraint (1) and the initial condition $x(t_0) = x_0$. Herein we consider the *infinite horizon* problem, $i.e.$ $t_f \to \infty$, and the cost functional

$$J(x(t), u(t)) = \frac{1}{2}\int_0^{\infty} (q(x) + u^T u)dt\,, \tag{3}$$

where $q : \mathbb{R}^n \to \mathbb{R}_+$ is positive semi-definite, subject to the dynamical constraint (1), the initial condition $x(t_0) = x_0$ and we replace the terminal condition on $x(t_f)$ with the requirement that the zero equilibrium of the closed-loop system be locally asymptotically stable.

A possible control law design methodology hinges upon the solvability of a partial differential equation, $i.e.$ the Hamilton-Jacobi-Bellman (HJB) equation[1]

$$\min_u \left\{ V_x(f(x) + g(x)u) + \frac{1}{2}q(x) + \frac{1}{2}u^T u \right\} = 0\,. \tag{4}$$

The major drawback of this approach is that an explicit solution of the HJB equation may be impossible to compute in practical applications.

To cope with this issue the problem of finding an approximate solution of (4) has been extensively addressed in recent years [2], [3], [4], [6], [7], [8], [9], [11], [12]. Sufficient conditions that guarantee the convergence of the Galerkin approximation of the HJB equation over a compact set containing the origin are given in [3]. In [6] conditions for the existence of a local solution of the HJB equation, for a parameterized family of infinite horizon optimal control problems are given. In [7] under the assumption of stabilizability of the nonlinear system, which is supposed to be real analytic about the origin, an optimal stabilizing control defined as the sum of a linear control law, that solves the corresponding linearized problem, and a convergent power series about the origin beginning with terms of second order is proposed. In [8] it is shown that the solution of the HJB equation is the eigenfunction, relative to the zero eigenvalue, of the semigroup, which is max-plus linear, corresponding to the HJB equation and a discrete-time approximation of the semigroup guarantees the convergence of the approximate solution to the actual one.

A successive approximation approach to design an optimal controller for a class of nonlinear systems with a quadratic performance index is presented in [11]. The methodology allows to transform the nonlinear optimal control problem into a sequence of non-homogeneous linear two-point boundary value problems. The optimal control law consists of a linear feedback term and a nonlinear compensation term which is the limit of an adjoint vector sequence. The local solution proposed in [12] hinges upon the technique of *apparent linearization* and the repeated computation of the steady-state solution of a Riccati equation. Since the assumption of differentiability of the solution of the HJB equation is restrictive, in recent years a large effort has been devoted to

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[1] In what follows we use the notation V_x to denote the gradient of the function V with respect to the vector x.

avoid this hypothesis, interpreting the HJB equation in the viscosity sense [2], [4], [9].

The main contribution of this paper is a method to construct dynamically an exact solution of a (modified) HJB equation, $i.e.$ considering the immersion of the system (1) into an *augmented* system, without actually solving any partial differential equation, see also [10].

The rest of the article is organized as follows. In the next section the definition of the problem studied in the paper is given. In Section III we present a dynamic optimal control law which guarantees asymptotic stability of the origin of the closed-loop system and minimizes a meaningful cost, which upperbounds the original cost (3). Furthermore, in the same section it is shown that the result can be improved yielding a dynamic control law that minimizes the *extra-cost*. In Section IV we show that the proposed approach, when applied to linear time-invariant systems, provides the standard optimal state feedback. Finally in the last two sections a case study is presented and conclusions are drawn, respectively.

II. Hamilton-Jacobi-Bellman equation

Consider system (1) and the following assumption.

Assumption 1: $f \in \mathcal{C}^1$ is such that $f(0) = 0$, $i.e.$ $x = 0$ is an equilibrium point for the system (1) when $u(t) = 0$ for all $t \geq 0$. Hence $f(x) = F(x)x$, for some, possibly not unique, continuous mapping $F(x) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$. Additionally suppose that the nonlinear system (1) is stabilizable, $i.e.$ there exists a smooth positive definite function $V(x)$ such that $\inf_u [V_x f(x) + V_x g(x)u] < 0$ for all $x \neq 0$ together with the assumption of *small control property*, and that the nonlinear system (1) with output $q(x)$ is zero-state detectable.

Problem 1: Consider system (1), under Assumption 1, and the cost functional (3). The regional dynamic optimal control problem consists in determining an integer[2] $\tilde{n} \geq 0$, a dynamic control law of the form

$$\begin{aligned} \dot{\xi} &= \alpha(x, \xi)\,, \\ u &= \beta(x, \xi)\,, \end{aligned} \tag{5}$$

with $\xi(t) \in \mathbb{R}^{\tilde{n}}$, $\alpha : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}^{\tilde{n}}$, $\beta : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}^m$ and a set $\bar{\Omega} \subset \mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ containing the origin of $\mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ such that the closed-loop system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\beta(x, \xi)\,, \\ \dot{\xi} &= \alpha(x, \xi)\,, \end{aligned} \tag{6}$$

has the following properties.

(i) The zero equilibrium of the system (6) is asymptotically stable with region of attraction containing $\bar{\Omega}$.
(ii) For any $\bar{u}(t)$ and any (x_0, ξ_0) such that the trajectory of the system (6) remain in $\bar{\Omega}$

$$J((x_0, \xi_0), \beta) \leq J((x_0, \xi_0), \bar{u})\,.$$

$\diamond$

[2]If $\tilde{n} = 0$ the term dynamic is abused.

The solution of the HJB equation (4), if it exists, is the *value function* of the optimal control problem, $i.e.$ it is a function that associates to every point x_0 in the state space the optimal cost of the trajectory of system (1) with $x(0) = x_0$, $i.e.$

$$V(x_0) = \min_u \frac{1}{2} \int_0^\infty (q(x) + u^T u)dt\,. \tag{7}$$

Knowledge of the value function on the entire state space allows to determine the minimizing input for all initial conditions. Note that if the optimal control problem is meaningful then necessarily the cost imposed to the state of the system and the control law must be positive, hence $V(x)$ must be positive definite. It is easy to check that the minimum of equation (4) with respect to u is attained for

$$u_o = -g(x)^T V_x^T\,. \tag{8}$$

Thus, if we are able to solve analytically the partial differential equation

$$V_x f(x) - \frac{1}{2} V_x g(x) g(x)^T V_x^T + \frac{1}{2} q(x) = 0\,, \tag{9}$$

we can design the optimal control policy given by (8), which is a solution of the dynamic optimal control problem with $\tilde{n} = 0$. Thus, rather than just searching for the control minimizing (3) and for the value of $V(x(t))$ for various x_0, the problem is approached by considering the evaluation of $V(x(t))$ for all $x(t)$ and the associated optimal policy.

Finally recall that in the linear case the solution of the optimal control problem is a linear static state feedback of the form $u = -B^T \bar{P} x$, where $\bar{P}$ is the symmetric positive definite solution of the algebraic Riccati equation

$$\bar{P}A + A^T \bar{P} - \bar{P}BB^T\bar{P} + Q = 0\,, \tag{10}$$

with

$$A \triangleq \frac{\partial f}{\partial x}\Big|_{x=0} \qquad Q \triangleq \frac{\partial^2 q}{\partial x^2}\Big|_{x=0} \qquad B \triangleq g(x)\Big|_{x=0}\,.$$

III. Dynamic Optimal Control Law

Before stating the main result a preliminary lemma is presented.

Lemma 1: [1] Let M be an $n \times n$ symmetric matrix and C an $m \times n$ matrix of rank m, where $m < n$. Let Z denote a basis for the null space of C.

(i) If $Z^T M Z$ is positive semidefinite and singular, then there exists a finite $\bar{k} \geq 0$ such that $M + kC^T C$ is positive semidefinite for all $k \geq \bar{k}$, if and only if $Ker(Z^T M Z) = Ker(MZ)$. In this case, $M + kC^T C$ is singular for all k.
(ii) $Z^T M Z$ is positive definite if and only if there exists a finite $\bar{k} \geq 0$ such that $M + kC^T C$ is positive definite for all $k \geq \bar{k}$.

$\diamond$

Consider now equation (9) and suppose that it can be solved *algebraically*, as detailed in the following definition.

Definition 1: Consider system (1), under Assumption 1. A $\mathcal{C}^1$ mapping $P(x) : \mathbb{R}^n \to \mathbb{R}^{1\times n}$, zero at zero, is said to be an *algebraic $\bar{P}$ solution* of (9) if there exists $\sigma(x) \triangleq x^T \Sigma(x) x > 0$, for all $x \in \mathbb{R}^n \setminus \{0\}$, with $\Sigma(x) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\Sigma(0) = 0$, such that

$$P(x)f(x) + \frac{1}{2}q(x) - \frac{1}{2}P(x)g(x)g(x)^T P(x)^T + \sigma(x) = 0\,, \tag{11}$$

and $P(x)$ is tangent in $x = 0$ to the symmetric positive definite solution of (10), $i.e.$

$$\frac{\partial P(x)^T}{\partial x}\Big|_{x=0} = \bar{P}\,.$$

◇

Remark 1: Let $P(x) = [P_1(x), ..., P_n(x)]$, with $P_i(x) : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, ..., n$, be an *algebraic $\bar{P}$ solution* of (9). Then there exists $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(x) = P(x)\,,$$

if and only if

$$\frac{\partial P_i}{\partial x_j}(x) = \frac{\partial P_j}{\partial x_i}(x)\,, \tag{12}$$

for all x and $i, j = 1, 2, ..., n$. Obviously, since an arbitrary *algebraic* solution of the HJB inequality is selected, the mapping $P(x)$ may not satisfy condition (12).

Using the *algebraic $\bar{P}$ solution* $P(x)$, define the function

$$V(x, \xi) = P(\xi)x + \frac{1}{2}\|x - \xi\|_R^2\,, \tag{13}$$

with $\xi(t) \in \mathbb{R}^n$ and $R = R^T \in \mathbb{R}^{n\times n}$ positive definite. Note that $\|v\|_R^2$ denotes the Euclidean norm of the vector v weighed by the matrix R, $i.e.$ $\|v\|_R^2 = v^T Rv$. To provide a concise statement of the main result yielding a solution to the regional dynamic optimal control problem, define

$$\Delta(x, \xi) = (R - \Phi(x, \xi))\Lambda(\xi)^T\,, \tag{14}$$

with $\Lambda(\xi) = \Psi(\xi)R^{-1}$, where $\Phi(x, \xi) \in \mathbb{R}^{n\times n}$ is a continuous mapping such that

$$P(x) - P(\xi) = (x - \xi)^T \Phi(x, \xi)^T\,, \tag{15}$$

$\Psi(\xi) \in \mathbb{R}^{n\times n}$ is the Jacobian matrix of the mapping $P(\xi)$,

$$A_{cl}(x) = F(x) - g(x)g(x)^T N(x)\,, \tag{16}$$

with $N(x)$ such that $P(x) = x^T N(x)^T$. Note that the vector field $A_{cl}(x)x$ describes the closed-loop system when the *algebraic* feedback $u = -g(x)^T P(x)^T$ is applied.

Theorem 1: Consider system (1), under Assumption 1 and the cost defined in (3). Let $P(x)$ be an *algebraic $\bar{P}$ solution* of (9). Let the matrix $R > 0$ be such that $V(x, \xi)$ is positive definite in a set $\Omega \subseteq \mathbb{R}^{2n}$ containing the origin and such that

$$\frac{1}{2}A_{cl}(x)^T \Delta + \frac{1}{2}\Delta^T A_{cl}(x) < \Sigma(x) + \frac{1}{2}\Delta^T g(x)g(x)^T \Delta\,, \tag{17}$$

for all $(x, \xi) \in \Omega \setminus \{0\}$.

Then there exists $\bar{k}$ such that for all $k \geq \bar{k}$ the function $V(x, \xi) > 0$, defined as in (13), satisfies for all $(x, \xi) \in \Omega$ the Hamilton-Jacobi-Bellman inequality

$$\begin{aligned}\mathcal{HJB}(x, \xi) &\triangleq V_x(x, \xi)f(x) + V_\xi(x, \xi)\dot{\xi} \\ &+ \frac{1}{2}q(x) - \frac{1}{2}V_x(x, \xi)g(x)g(x)^T V_x(x, \xi)^T \leq 0\,,\end{aligned} \tag{18}$$

with

$$\dot{\xi} = -kV_\xi(x, \xi)^T = -k(\Psi(\xi)^T x - R(x - \xi))\,. \tag{19}$$

Hence

$$\begin{aligned}\dot{\xi} &= -k(\Psi(\xi)^T x - R(x - \xi))\,, \\ u &= -g(x)^T \left[P(x)^T + (R - \Phi(x, \xi))(x - \xi)\right]\,,\end{aligned} \tag{20}$$

solves the regional dynamic optimal control problem with instantaneous cost $L(x, \xi, u) = q(x) + h(x, \xi) + u^T u$ where $h(x, \xi) \geq 0$ is such that

$$\mathcal{HJB}(x, \xi) + \frac{1}{2}h(x, \xi) = 0\,.$$

◇

Remark 2: Consider V as in (13) and note that there exist a compact set $\Omega \subseteq \mathbb{R}^{2n}$ containing the origin and a positive definite matrix $\bar{R}$ such that for all $R \geq \bar{R}$ the function V is positive definite for all $(x, \xi) \in \Omega \subseteq \mathbb{R}^{2n}$. In fact since $P(x)$ is tangent in $x = 0$ to the solution of the algebraic Riccati equation then the function $P(x)x$ is, locally around the origin, quadratic and moreover has a local minimum for $x = 0$[3]. Hence, the existence of $\bar{R}$ can be proved noting that the function $P(\xi)x$ is (locally) quadratic and, restricted to the manifold $\mathcal{M} = \{\xi \in \mathbb{R}^n : \xi = x\}$, is positive definite for all $x \neq 0$ in Ω.

Remark 3: The zero-state detectability property, assumed for the nonlinear system $\dot{x} = f(x) + g(x)u$ with output $q(x)$, holds also for the *extended* system (6), with $\alpha(x, \xi)$ and $u(x, \xi)$ defined in (20), with respect to the same output. To prove the claim, consider system (6) and note that $u(x(t), \xi(t)) = 0$, $q(x(t)) = 0$ for all $t \geq 0$ imply, by Assumption 1, that x asymptotically converges to the origin while ξ belongs, by (18), to the compact set $\{(x, \xi) : V(x, \xi) \leq V(x(0), \xi(0))\}$ for all $t \geq 0$. Therefore, system (6) *reduces*, for t sufficiently large, to the system $\dot{\xi} = -kR\xi$, which has an asymptotically stable equilibrium at the origin. Concluding, $u(x(t), \xi(t)) = 0$ and $q(x(t)) = 0$ for all $t \geq 0$ imply that (x, ξ) asymptotically converges to the origin, hence the claim.

Remark 4: By the condition (18), V is a non-strict Lyapunov function for the closed-loop system (6). In fact, $V(x, \xi) > 0$ for all $(x, \xi) \in \Omega \setminus \{0\}$ and $\dot{V} \leq 0$, hence by LaSalle's invariance principle and zero-state detectability, the feedback (20) asymptotically stabilizes the zero equilibrium of the closed-loop system.

[3]This can be easily proved considering that the first-order derivative of the function is zero in $x = 0$ and $(P(x)x)_x = 2\bar{P} > 0$.

Remark 5: An *algebraic $\bar{P}$ solution* can be easily determined since the equation (11) does not involve partial derivatives of the unknown function to be determined, $i.e.$ the difficulty of solving partial differential equations has been removed. The control law defined in Theorem 1 is optimal for the system (1) with respect to a cost functional given by the sum of the original cost $q(x) + u^T u$ and the positive definite function $h(x, \xi)$.

Remark 6: The problem solved herein is intrinsecally different from the so-called *inverse optimal control problem* [5], where it is shown that optimality with respect to any meaningful cost functional guarantees several robustness properties of the closed-loop system. In fact, herein optimality of the control law is ensured with respect to a cost functional which upperbounds the original one, $i.e.$ the optimal trajectories of the control and the state variables are determined with respect to the original cost and an *extra-cost*, which is necessarily paid to relax the gradient condition (12).

In addition the control law can be modified in order to minimize the *extra-cost* imposed on the state. The minimization can be achieved shaping the cost function to approximate the original cost and selecting the initial condition of the dynamic extension, as detailed in the following result.

To state the result define $C(\xi) = [\Psi(\xi)^T \; -R] \in \mathbb{R}^{n\times 2n}$, and note that $C(\xi)$ has constant rank n, since R is positive definite, and

$$M(x,\xi) = \begin{bmatrix} \Sigma(x) & \Gamma_1(x,\xi) \\ \Gamma_1(x,\xi)^T & \Gamma_2(x,\xi) \end{bmatrix},$$

where $\Gamma_1(x,\xi) = -\frac{1}{2}A_{cl}(x)^T(R-\Phi(x,\xi))$ and $\Gamma_2(x,\xi) = \frac{1}{2}(R-\Phi(x,\xi))^T g(x)g(x)^T(R-\Phi(x,\xi))$. Note that the null space of $C(\xi)$ can be written as the space spanned by the columns of the matrix $Z(\xi)$, where

$$Z(\xi) = Ker(C(\xi)) = \begin{bmatrix} I \\ R^{-1}\Psi(\xi)^T \end{bmatrix}. \qquad (21)$$

Finally, consider the change of coordinates $[x^T \; (x - \xi)^T]^T = T(x,\xi)y$, with $y = [y_1^T \; y_2^T]^T \in \mathbb{R}^{2n}$ and

$$T(x,\xi) = \begin{bmatrix} Z^T \\ C \end{bmatrix} \begin{bmatrix} I_n & -(Z^TMZ)^{-1}Z^TMC^T \\ 0_n & I_n \end{bmatrix}.$$

Note that the matrix $T(x,\xi) \in \mathbb{R}^{2n\times 2n}$ is well-defined and non-singular by construction if the condition (17) is satisfied.

Theorem 2: Consider system (1), under Assumption 1 and the cost defined in (3). Let $P(x)$ be an *algebraic $\bar{P}$ solution* of (9). Let the matrix $R > 0$ be such that $V(x,\xi)$ is positive definite in a set $\Omega \subseteq \mathbb{R}^{2n}$ containing the origin and such that the condition

$$0 < \Sigma(x) + \frac{1}{2}\Delta^T g(x)g(x)^T\Delta - \frac{1}{2}\left(A_{cl}(x)^T\Delta + \Delta^T A_{cl}(x)\right) \leq \varepsilon I$$

is satisfied for all $(x,\xi) \in \Omega \setminus \{0\}$ and for some $\varepsilon \in \mathbb{R}_+$. Let $K(\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be defined as

$$K(x,\xi) = (CC^T)^{-1}\Pi(x,\xi)(CC^T)^{-1}, \qquad (22)$$

where $\Pi = CMZ\left(Z^TMZ\right)^{-1}Z^TMC^T - CMC^T$ and suppose that $K(x,\xi)$ is continuous in $(x,\xi) = (0,0)$. Then the control

$$\begin{aligned} \dot{\xi} &= -K(x,\xi)(\Psi(\xi)^T x - R(x-\xi)), \\ u &= g(x)^T\left[P(x)^T + (R - \Phi(x,\xi))(x-\xi)\right], \end{aligned} \qquad (23)$$

solves the regional dynamic optimal control problem with instantaneous cost $L(x,\xi,u) = q(x) + \bar{h}(x,\xi) + u^T u$ where $0 \leq \bar{h}(x,\xi) \leq \varepsilon y_1^T y_1$, for all $(x,\xi) \in \Omega$.

◇

Note that, by definition of *value function*, $V(x(0),\xi(0))$ is the optimal value of the cost minimized by the dynamic control law defined in (23) for the system (1) initialized in $(x(0),\xi(0))$. Therefore to minimize the optimal cost, for a given initial condition $\bar{x}(0)$ of system (1), it is possible to select the initial condition of the dynamic extension $\xi(0)$ as

$$\xi_{\bar{x}(0)}(0) = \arg\min_{\xi} V(\bar{x}(0),\xi).$$

IV. Linear Systems

Consider a linear time-invariant system described by equations of the form

$$\dot{x} = Ax + Bu, \qquad (24)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and the quadratic cost

$$J(x(t),u(t)) = \frac{1}{2}\int_0^\infty (x^TQx + u^Tu)dt.$$

The following statement gives the counterpart of the results presented in Theorem 1 to linear systems. It is interesting to note that the methodology proposed in Theorem 1 yields the standard solution of the linear quadratic infinite horizon optimal control problem.

Proposition 1: Consider system (24). Suppose that there exists a positive definite matrix $\bar{P} \in \mathbb{R}^{n\times n}$ such that[4]

$$\bar{P}A + A^T\bar{P} - \bar{P}BB^T\bar{P} + Q = 0. \qquad (25)$$

Then there exists R such that the conditions in Theorem 1 are trivially satisfied for all $(x,\xi) \in \mathbb{R}^{2n}$. Moreover the control law (20) reduces to

$$\begin{aligned} \dot{\xi} &= -k\bar{P}\xi, \\ u &= -B^T\bar{P}x. \end{aligned} \qquad (26)$$

◇

Note that with $\xi(0) = 0$ the static state feedback optimal control law is recovered.

[4]Since $\Sigma(0) = 0$, the equation can be considered as the linear condition equivalent to the condition (11) in Theorem 1.

Remark 7: The result obtained in the linear setting suggests a possible choice for the matrix R in the nonlinear case. In fact, selecting $R = \Phi(0,0) > 0$, with Φ defined as in (15), yields $\Delta(0,0) = 0$, hence, since Δ is continuous, it is sufficiently small in a neighborhood of the origin. Moreover, assume additionally that $\Sigma(0) = \bar{\Sigma} > 0$ in the definition of *algebraic* $\bar{P}$ *solution*. Then, by continuity of the left-hand side of inequality (17), there exists a non-empty subset $\hat{\Omega} \subset \mathbb{R}^{2n}$ containing the origin such that the condition (17) is satisfied for all $(x, \xi) \in \hat{\Omega}$.

V. Example: The Van Der Pol Oscillator

The single-input, single-output, nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= -x_1 - \mu(1 - x_1^2)x_2 + x_1 u\,, \end{aligned} \tag{27}$$

known as the Van der Pol oscillator, with $x(t) = (x_1(t)\,, x_2(t))^T \in \mathbb{R}^2$ and $u(t) \in \mathbb{R}$, is a non-conservative oscillator with nonlinear damping. The parameter μ describes the strength of the damping effect and in the following is selected as $\mu = 0.5$, hence the oscillator has a stable but linearly uncontrollable equilibrium at the origin and an unstable limit cycle. Define the positive definite cost $q(x) = x_2^2$, $i.e.$ the control action minimizes the *speed* of the oscillator together with the control effort. Note that the solution of the HJB equation for system (27) with the given cost can be explicitly determined, namely $V_o(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, and the resulting optimal control is $u_o = -x_1 x_2$. Since the linearization of the nonlinear system around the origin is given by $\dot{x} = Ax$, $i.e.$ it is not affected by u, with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{1}{2} \end{pmatrix},$$

which is Hurwitz, then the solution of the algebraic Riccati equation (10) corresponding to the linearized problem yields the matrix $\bar{P} = I$. The selection

$$\Sigma(x) = \epsilon \begin{pmatrix} \frac{1}{2}x_2^2 & 0 \\ 0 & \frac{1}{2}x_1^2 \end{pmatrix}$$

with $\epsilon > 0$, is such that $P(x) \in \mathbb{R}^{1\times 2}$ defined as

$$P(x) = [x_1(1 - \epsilon x_1 x_2)\,, x_2]$$

is an *algebraic* $\bar{P}$ *solution* of the Hamilton-Jacobi-Bellman equation. Note that for this solution the condition (12) is not satisfied, hence $P(x)$ is not an exact differential. However, Theorem 1 guarantees the existence of a compact set $\Omega \subset \mathbb{R}^{2n}$, and a matrix R such that the function $V(x, \xi)$ is positive definite for all $(x, \xi) \in \Omega$. Let $R = diag(\alpha, \alpha)$ with $\alpha > 1$, then the function $V(x, \xi)$ is locally positive definite in the set $\Omega_1 = \{ (x, \xi) \in \mathbb{R}^4 : |\xi_1| \leq 1\}$ and moreover the condition (17) is strictly satisfied for all $(x, \xi) \in \Omega \setminus \{0\}$, where $\Omega = \Omega_1 \cap \Omega_2 = \{(x, \xi) \in \mathbb{R}^4 : \mathcal{X}(x, \xi) > 0\}$, $\mathcal{X}$ being a known continuous function, hence it is possible to determine a value $\bar{k} \geq 0$ such that the *extended* Hamilton-Jacobi-Bellman equation holds. Since $R - \Phi(x, \xi)$ is not zero in $Im(g(x))$ then the dynamic control law $u(x, \xi) = -\alpha x_1 x_2 + (\alpha - 1)x_1 \xi_2$ solves the dynamic optimal control problem with $L(x, \xi, u) = x_2^2 + u^2 + h(x, \xi)$, where $h(x, \xi) \geq 0$ is defined as in Theorem 1. (see also Figure 1, where the functions $L_o(x) = x_2^2$ and $L_d(x) = x_2^2 + h(x, \xi_{\bar{x}(0)})$ are displayed, bottom and top surface respectively)

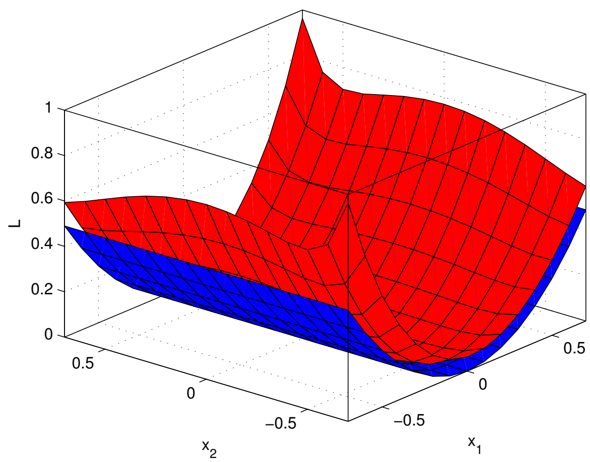

Fig. 1. Instantaneous cost imposed on the state of the original problem, $i.e$ $L(x) = x_2^2$ (bottom surface) and of the modified problem, namely $L(x) = x_2^2 + h(x, \xi_{\bar{x}(0)})$ (top surface), respectively.

In the first simulation we let $k = 2$ and $\alpha = 1.4$ and we have compared the static optimal feedback, the dynamic optimal control law (20) and the optimal solution of the linearized problem, namely $u_l(t) \equiv 0$. The top graph of Figure 2 shows the time histories of the state of the system (27) when the optimal static feedback (solid line) and the dynamic control law $u(x, \xi)$ (dashed line) are applied, respectively. The behavior of the state of the system driven by the dynamic control law is closer to the optimal behavior than the one obtained with the optimal solution of the linearized problem (dash-dotted line). In the middle graph of Figure 2 the time histories of the optimal cost (solid line) and the cost yielded by the dynamic solution (dashed line) are displayed, together with the cost resulting when the zero input is applied (dash-dotted line). (Note that this is the optimal control for the linearized problem.) The bottom graph of Figure 2 shows the time histories of the optimal static feedback (solid line) and the dynamic control law (20) (dashed line), respectively. It is worth noting that even if the relative error between the optimal cost and the dynamic cost is around 0.18 the actual solution of Problem 1, $i.e.$ the time evolution of the minimizing control law and the state of the system, is close to the optimal solution.

In the second simulation a comparison between the optimal values obtained applying the optimal static feedback and the dynamic control law is performed for different values of the initial condition of system (27). The values of $V_o(x(0))$ (bottom surface) and of $V(x(0), \xi(0))$ for $\xi(0) = 0$ (top surface) or $\xi(0) = \xi_{\bar{x}(0)}(0)$ for each $\bar{x}(0)$ (middle surface) are depicted as functions of $x(0)$. Note that $\xi_{\bar{x}(0)}(0)$ can be determined analytically. Finally, Figure 4 shows the relative

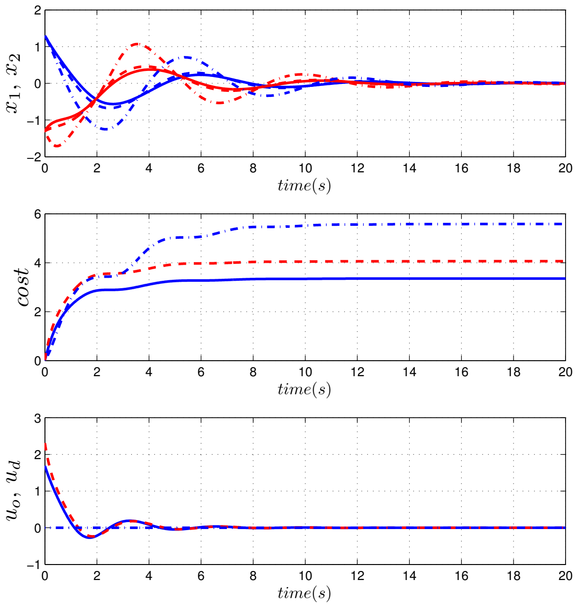

Fig. 2. Top graph: time histories of the state of the state of the system (27) driven by the optimal static feedback (solid lines) and by the dynamic control law $u(x,\xi)$ (dashed lines). Middle graph: time histories of the optimal cost functional (solid line), the dynamic cost functional (dashed line) and the cost functional for the optimal solution of the linearized system (dash-dotted line), respectively. Bottom graph: time histories of the static optimal feedback (solid line) and the dynamic control law (dashed line), respectively.

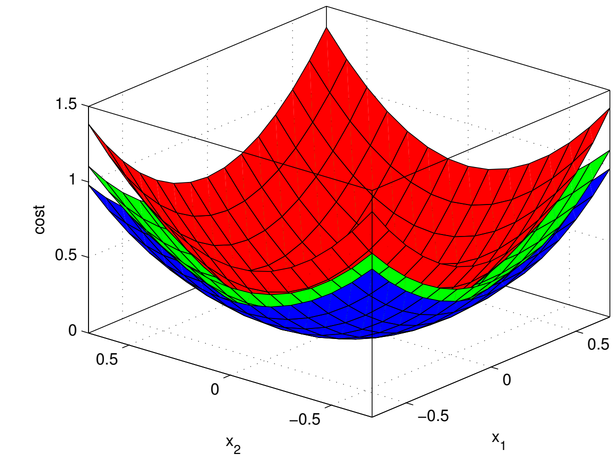

Fig. 3. Values of the costs yielded by the optimal static feedback (bottom surface) and by the dynamic control law, considering the optimal value of $\xi(0)$ for each $x(0)$, (middle surface) and for constant $\xi(0) = 0$ (top surface).

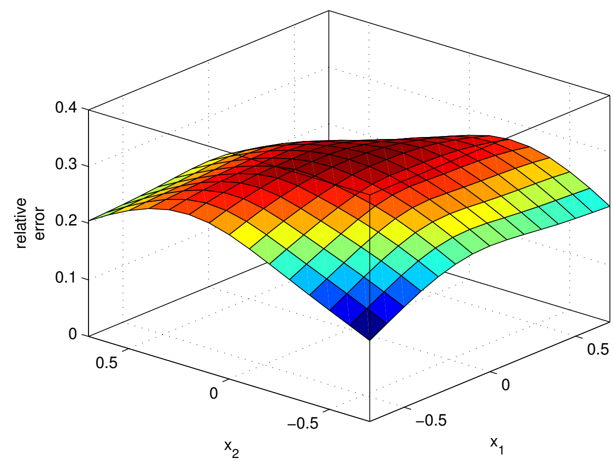

Fig. 4. Relative error between the optimal cost, $V_l(x(0))$, and the dynamic optimal cost, $V(x(0), \xi_{\bar{x}(0)}(0))$.

error between the optimal cost, $V_o(x(0))$, and the dynamic optimal cost, $V(x(0), \xi_{\bar{x}(0)}(0))$. Note that the relative error ranges between 0.1 and 0.25.

VI. Conclusions

In this paper we have studied the optimal control problem for nonlinear systems. We have shown that the explicit solution of the HJB equation is not needed provided an additional cost is *paid*. The methodology yields a dynamic control law that stabilizes the equilibrium of the closed-loop system and minimizes a meaningful cost functional. The latter is given by the sum of the original cost and an *extra-cost*, that can be minimized with a proper selection of the initial condition of the dynamic controller.

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